STATISTICAL THEORY OF THE NONUNIFORM TURBULENCE OF AN INCOMPRESSIBLE FLUID

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The nonuniform turbulence problem is treated by a statistical approach based on the use of a finite number of equations for the higher-order single-point correlations. Additional differential equations are derived for the "unknown" moments in the single-point correlation equations. The equations are closed by means of approximate expressions for the anisotropic two-point correlation tensors for near points. A closed system of seven tensorial differential equations is given, describing the variation of the fundamental characteristics of nonuniform turbulence.

Attempts have been made recently [1] to formulate a theory of nonuniform turbulence on the basis of a finite number of equations for the higher-order correlations. This approach to the problem has undoubtedly received its fullest treatment in [2-13]. In the latter, the equations for the single-point correlations are closed by the introduction of certain phenomenological hypotheses based on the formal analogy between turbulent and molecular momentum transfer. The larger the number of correlation equations analyzed in this case, the larger must be the number of hypotheses, so that it becomes necessary to determine a large number of dimensionless empirical coefficients.

In the present article we endeavor to set forth a statistical description of the dynamics of nonuniform turbulence in an incompressible fluid on the basis of the equations for the single- and two-point correlations of the fluctuation variables. The fundamental system of differential equations for the velocity correlations is closed, not by the use of phenomenological hypotheses, but by means of additional differential equations for the "unknown" correlations and expressions for the anisotropic two-point correlation tensors for closely spaced points.

The fundamental equations are as follows:

incompressibility equations:

$$\frac{\partial U_k}{\partial x_k}=0, \tag{1}$$

Reynolds equations:

$$\frac{\partial U_i}{\partial \tau}+U_k\frac{\partial U_i}{\partial x_k}+\frac{\partial}{\partial x_k}\overline{u_iu_k}=-\frac{1}{\rho}\cdot\frac{\partial P}{\partial x_i}+\nu\Delta_x U_i, \tag{2}$$

double single-point correlation equations [1]:

$$\frac{\partial}{\partial \tau}\overline{u_iu_j}+U_k\frac{\partial}{\partial x_k}\overline{u_iu_j}+\overline{u_iu_k}\frac{\partial U_j}{\partial x_k}+\overline{u_ju_k}\frac{\partial U_i}{\partial x_k}-\nu\Delta_x\overline{u_iu_j}$$
$$+\frac{\partial}{\partial x_k}\overline{u_iu_ju_k}+\frac{1}{\rho}\overline{\left(u_i\frac{\partial p}{\partial x_j}+u_j\frac{\partial p}{\partial x_i}\right)}+2\nu\overline{\frac{\partial u_i}{\partial x_k}\cdot\frac{\partial u_j}{\partial x_k}}=0 \tag{3}$$

triple single-point correlation equations [7]:

$$\frac{\partial}{\partial \tau}\overline{u_iu_ju_k}+U_l\frac{\partial}{\partial x_l}\overline{u_iu_ju_k}+\overline{u_iu_ju_l}\frac{\partial U_k}{\partial x_l}+\overline{u_ju_ku_l}\frac{\partial U_i}{\partial x_l}$$

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$$+\overline{u_iu_ku_l}\frac{\partial U_j}{\partial x_l}-\nu\Delta_x\overline{u_iu_ju_k}-\left(\overline{u_iu_j}\cdot\frac{\partial}{\partial x_l}\overline{u_ku_l}+\overline{u_ju_k}\frac{\partial}{\partial x_l}\overline{u_iu_l}\right.$$

$$\left.+\overline{u_iu_k}\frac{\partial}{\partial x_l}\overline{u_ju_l}\right)+\frac{\partial}{\partial x_l}\overline{u_iu_ju_ku_l}$$

$$+\underline{\frac{1}{\rho}\left(\overline{u_iu_j\frac{\partial p}{\partial x_k}}+\overline{u_ju_k\frac{\partial p}{\partial x_i}}+\overline{u_iu_k\frac{\partial p}{\partial x_j}}\right)}$$

$$+\underline{2\nu\left(\overline{u_i\frac{\partial u_j}{\partial x_l}\cdot\frac{\partial u_k}{\partial x_l}}+\overline{u_j\frac{\partial u_i}{\partial x_l}\cdot\frac{\partial u_k}{\partial x_l}}+\overline{u_k\frac{\partial u_j}{\partial x_l}\cdot\frac{\partial u_i}{\partial x_l}}\right)}=0. \quad (4)$$

We adopt the following hypothesis of Millionshchikov [14] as the fundamental hypothesis limiting the number of correlation equations:

$$\overline{u_iu_ju_ku_l}=\overline{u_iu_j}\;\overline{u_ku_l}+\overline{u_iu_k}\;\overline{u_ju_l}+\overline{u_iu_l}\;\overline{u_ju_k}. \quad (5)$$

It is well known [1] that hypothesis (5) is strictly valid only for fields having a Gaussian density function. However, as shown by numerous experimental data, it can be regarded as fully satisfactory for almost-isotropic flows [15-19] and for strongly nonuniform flows [20-23].

Despite the fact that hypothesis (5) can be used to produce a situation in which the number of velocity correlations and the number of equations describing them are the same, the inclusion in the nth-order correlation equations of equal-order correlations containing derivatives of the velocities and pressure [the underscored terms in Eqs. (3)-(4)] prevents the closure of this fundamental system of equations.

For the investigation of the above-indicated unknown correlations it is advisable to represent them as functions of the two-point correlations, introducing the new coordinate system [24]

$$\zeta_k=(x_k)_B-(x_k)_A,\;(x_k)_{AB}=\frac{1}{2}[(x_k)_A+(x_k)_B], \quad (6)$$

which reflects the dependence of the two-point correlations on the distance between two given points A and B and on the positions of those points in the flow field. Then the correlation characterizing the dissipation of fluctuation kinetic energy in Eq. (3) can be written in the form

$$\overline{\frac{\partial u_i}{\partial x_k}\cdot\frac{\partial u_j}{\partial x_k}}=\frac{1}{4}[(\Delta_x)_{AB}\overline{u_iu_j'}]_0-(\Delta_\zeta\overline{u_iu_j'})_0. \quad (7)$$

We now derive a differential equation describing the variation of the tensor $(\Delta_\zeta\overline{u_iu_j'})_0$ in the nonuniform turbulence field. Our starting point is the dynamical equation for the two-point velocity correlation in nonuniform turbulence; in the coordinates (6) this equation takes the form [24]

$$\frac{\partial}{\partial\tau}\overline{u_iu_j'}+\overline{u_ku_j'}\left(\frac{\partial U_i}{\partial x_k}\right)_A+\overline{u_iu_k'}\left(\frac{\partial U_j}{\partial x_k}\right)_B+\frac{1}{2}[(U_k)_A+(U_k)_B]$$

$$\times\left(\frac{\partial}{\partial x_k}\right)_{AB}\overline{u_iu_j'}+[(U_k)_B-(U_k)_A]\frac{\partial}{\partial\zeta_k}\overline{u_iu_j'}$$

$$+\frac{1}{2}\left(\frac{\partial}{\partial x_k}\right)_{AB}(\overline{u_iu_k'u_j'}+\overline{u_iu_ku_j'})+\frac{\partial}{\partial\zeta_k}(\overline{u_iu_k'u_j'}-\overline{u_iu_ku_j'})$$

$$+\frac{1}{2\rho}\left[\left(\frac{\partial}{\partial x_i}\right)_{AB}\overline{pu_j'}+\left(\frac{\partial}{\partial x_j}\right)_{AB}\overline{u_ip'}\right]-\frac{1}{\rho}\left(\frac{\partial}{\partial\zeta_i}\overline{pu_j'}\right.$$

$$\left.-\frac{\partial}{\partial\zeta_j}\overline{u_ip'}\right)-\frac{1}{2}\nu(\Delta_x)_{AB}\overline{u_iu_j'}-2\nu\Delta_\zeta\overline{u_iu_j'}=0. \quad (8)$$

Performing the operation $[\Delta_\zeta(\;)]_0$ on Eq. (8) and carrying out some simple transformations associated with the introduction of the new coordinate system (6), we obtain the equation

$$\frac{\partial}{\partial\tau}(\Delta_\zeta\overline{u_iu_j'})_0+\frac{1}{4}\left(\Delta_x\frac{\partial U_i}{\partial x_k}\overline{u_ku_j}+\Delta_x\frac{\partial U_j}{\partial x_k}\overline{u_ku_i}\right)+\frac{\partial U_i}{\partial x_k}(\Delta_\zeta\overline{u_ku_j'})_0$$

$$+\frac{\partial U_j}{\partial x_k}(\Delta_\zeta\overline{u_iu_k'})_0-\frac{\partial^2U_i}{\partial x_k\partial x_l}\left(\frac{\partial}{\partial\zeta_l}\overline{u_ku_j'}\right)_0+\frac{\partial^2U_j}{\partial x_k\partial x_l}\left(\frac{\partial}{\partial\zeta_l}\overline{u_iu_k'}\right)_0$$

$$+\frac{1}{4}\Delta_x U_k\left[\left(\frac{\partial}{\partial x_k}\right)_{AB}\overline{u_i u_j'}\right]_0+U_k\left[\left(\frac{\partial}{\partial x_k}\right)_{AB}\Delta_\zeta\overline{u_i u_j'}\right]_0$$

$$+2\frac{\partial U_k}{\partial x_l}\left(\frac{\partial^2}{\partial\zeta_k\partial\zeta_l}\overline{u_i u_j'}\right)_0+\frac{1}{2}\left[\left(\frac{\partial}{\partial x_k}\right)_{AB}\Delta_\zeta(\overline{u_i u_j' u_k'}+\overline{u_i u_k u_j'})\right]_0$$

$$+\left[\Delta_\zeta\frac{\partial}{\partial\zeta_k}(\overline{u_i u_j' u_k'}-\overline{u_i u_k u_j'})\right]_0+\frac{1}{2\rho}\left[\left(\frac{\partial}{\partial x_i}\right)_{AB}\Delta_\zeta\overline{p u_j'}\right.$$

$$\left.+\left(\frac{\partial}{\partial x_j}\right)_{AB}\Delta_\zeta\overline{p' u_i}\right]_0-\frac{1}{\rho}\left[\Delta_\zeta\left(\frac{\partial}{\partial\zeta_i}\overline{p u_j'}-\frac{\partial}{\partial\zeta_j}\overline{p' u_i}\right)\right]_0$$

$$-\frac{1}{2}\nu\left[(\Delta_x)_{AB}\Delta_\zeta\overline{u_i u_j'}\right]_0-2\nu\left[\Delta_\zeta(\Delta_\zeta\overline{u_i u_j'})\right]_0=0. \quad (9)$$

The correlations containing pressure fluctuations in Eq. (3) may be represented as follows in the coordinate system (6):

$$\frac{1}{\rho}\left(\overline{u_i\frac{\partial p}{\partial x_j}}+\overline{u_j\frac{\partial p}{\partial x_i}}\right)=\frac{1}{2\rho}\left[\left(\frac{\partial}{\partial x_j}\right)_{AB}\overline{p u_i'}+\left(\frac{\partial}{\partial x_i}\right)_{AB}\overline{p u_j'}\right]_0-\frac{1}{\rho}\left[\frac{\partial}{\partial\zeta_j}\overline{p u_i'}+\frac{\partial}{\partial\zeta_i}\overline{p u_j'}\right]_0. \quad (10)$$

We next derive differential equations for the correlation $(1/\rho)\overline{pu_r'}$ entering into (10). As we know [1], the pressure fluctuations satisfy an equation of the Poisson type:

$$\frac{1}{\rho}\Delta_x p=-2\frac{\partial U_m}{\partial x_n}\cdot\frac{\partial u_n}{\partial x_m}-\frac{\partial^2}{\partial x_m\partial x_n}(u_m u_n-\overline{u_m u_n}). \quad (11)$$

Writing this equation at point A, multiplying by u_r', and averaging, we obtain

$$\frac{1}{\rho}(\Delta_x)_A\overline{p u_r'}=-2\left(\frac{\partial U_m}{\partial x_n}\right)_A\left(\frac{\partial}{\partial x_m}\right)_A\overline{u_n u_r'}-\left(\frac{\partial^2}{\partial x_m\partial x_n}\right)_A\overline{u_m u_n u_r'}.$$

Transforming to the new variables (6), we rewrite the latter equation in the form

$$\frac{1}{\rho}\left[\frac{1}{4}(\Delta_x)_{AB}\overline{p u_r'}+(\Delta_\zeta\overline{p u_r'})-\left(\frac{\partial}{\partial x_k}\right)_{AB}\left(\frac{\partial}{\partial\zeta_k}\overline{p u_r'}\right)\right]=-2\frac{\partial U_m}{\partial x_n}$$

$$\times\left[\frac{1}{2}\left(\frac{\partial}{\partial x_m}\right)_{AB}\overline{u_n u_r'}-\left(\frac{\partial}{\partial\zeta_m}\overline{u_n u_r'}\right)\right]-\frac{1}{4}\left[\left(\frac{\partial^2}{\partial x_m\partial x_n}\right)_{AB}\overline{u_m u_n u_r'}\right.$$

$$\left.-\frac{1}{2}\left(\frac{\partial}{\partial x_m}\right)_{AB}\left(\frac{\partial}{\partial\zeta_n}\overline{u_m u_n u_r'}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x_n}\right)_{AB}\left(\frac{\partial}{\partial\zeta_m}\overline{u_m u_n u_r'}\right)+\left(\frac{\partial^2}{\partial\zeta_m\partial\zeta_n}\overline{u_m u_n u_r'}\right)\right]. \quad (12)$$

Then we consider the unknown terms of the triple correlation equation (4). The correlations characterizing the dissipation of turbulence due to viscosity may be represented as follows in the variables (6):

$$2\left(\overline{u_i\frac{\partial u_j}{\partial x_l}\frac{\partial u_k}{\partial x_l}}+\overline{u_j\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}}+\overline{u_k\frac{\partial u_i}{\partial x_l}\frac{\partial u_j}{\partial x_l}}\right)$$

$$=\frac{1}{4}\left[(\Delta_x)_{AB}(\overline{u_i u_j u_k'}+\overline{u_i u_k u_j'}+\overline{u_j u_k u_i'})\right]_0-\left[\Delta_\zeta(\overline{u_i u_j u_k'}+\overline{u_i u_k u_j'}+\overline{u_j u_k u_i'})\right]_0, \quad (13)$$

and the correlations containing pressure fluctuations may be written in the form

$$\frac{1}{\rho}\left(\overline{u_i u_j\frac{\partial p}{\partial x_k}}+\overline{u_j u_k\frac{\partial p}{\partial x_i}}+\overline{u_i u_k\frac{\partial p}{\partial x_j}}\right)=\frac{1}{2\rho}\left[\left(\frac{\partial}{\partial x_i}\right)_{AB}\overline{p u_k' u_j'}\right.$$

$$\left.+\left(\frac{\partial}{\partial x_j}\right)_{AB}\overline{p u_i' u_k'}+\left(\frac{\partial}{\partial x_k}\right)_{AB}\overline{p u_i' u_j'}\right]_0-\frac{1}{\rho}\left[\frac{\partial}{\partial\zeta_i}\overline{p u_j' u_k'}+\frac{\partial}{\partial\zeta_j}\overline{p u_i' u_k'}+\frac{\partial}{\partial\zeta_k}\overline{p u_i' u_j'}\right]_0.$$

Now we derive a differential equation for the correlation $(1/\rho)\overline{pu_r u_s}$. To do so we make use of the Poisson equation (11) at point A. Multiplying this equation by $u_r'u_s'$ and averaging, we obtain

$$\frac{1}{\rho}(\Delta_x)_A\overline{p u_r' u_s'}=-2\left(\frac{\partial U_m}{\partial x_n}\right)_A\left(\frac{\partial}{\partial x_m}\right)_A\overline{u_n u_r' u_s'}-\left(\frac{\partial^2}{\partial x_m\partial x_n}\right)_A\overline{u_m u_n u_r' u_s'}+\overline{u_r' u_s'}\cdot\frac{\partial^2}{\partial x_m\partial x_n}\overline{u_m u_n}.$$

Transforming to the new variables (6), we rewrite this equation for $\zeta = 0$ in the form

$$\frac{1}{\rho}\left[\frac{1}{4}(\Delta_x)_{AB}\overline{pu'_r u'_s} + \Delta_\zeta \overline{pu'_r u'_s} - \left(\frac{\partial}{\partial x_l}\right)_{AB}\frac{\partial}{\partial \zeta_l}\overline{pu'_r u'_s}\right]_0$$

$$= -2\left(\frac{\partial U_m}{\partial x_n}\right)\left[\frac{1}{2}\left(\frac{\partial}{\partial x_m}\right)_{AB}\overline{u_n u'_r u'_s} - \frac{\partial}{\partial \zeta_m}\overline{u_n u'_r u'_s}\right]_0$$

$$-\left\{\left[\frac{1}{4}\left(\frac{\partial^2}{\partial x_m \partial x_n}\right)_{AB} - \frac{1}{2}\left(\frac{\partial}{\partial x_m}\right)_{AB}\frac{\partial}{\partial \zeta_n} - \frac{1}{2}\left(\frac{\partial}{\partial x_n}\right)_{AB}\frac{\partial}{\partial \zeta_m} + \frac{\partial^2}{\partial \zeta_m \partial \zeta_n}\right]\overline{u_m u_n u'_r u'_s}\right\}_0. \quad (14)$$

With (5) taken into account, Equations (1)-(4), (9), and (12)-(14) form a system of equations describing the behavior of the fundamental characteristics of nonuniform turbulence. These equations contain a series of unknown terms, which represent differential operators of the two-point correlations at the point $\zeta = 0$. Consequently, these terms can be determined if the corresponding two-point correlations for near points are known. We adopt as our criterion of "nearness" for two points the microscales of the corresponding correlations. Hence, if the correlation tensors are represented as expansions in power series on the dimensionless (referred to the appropriate microscale) coordinate ζ, then for closely spaced points we can limit the expansions to the first few terms. In the region between two near points the turbulence is assumed to be uniform. Thus, with the use of the two-point correlation tensors for two near points, in order to close the nonuniform turbulence equations with respect to the terms representing differential operators of the two-point correlations tensors we can invoke the concept of local uniformity, i.e., assume at any point of the flow field that the correlation tensors in the given terms are homogeneous (on the coordinate ζ), but vary in the space x_k.

The fundamental conditions that must be satisfied by the anisotropic two-point correlation tensors are as follows:

coincidence with the corresponding single-point correlations at $\zeta = 0$, i.e.,

$$(\overline{u_i u_j \ldots u'_m})_0 = \overline{u_i u_j \ldots u_m}, \quad (15)$$

coincidence with the corresponding isotropic tensors under isotropy, i.e.,

$$(\overline{u_i u_j \ldots u'_m})^* = Q_{ij\ldots m}, \quad (16)$$

coincidence of the differential operators of the anisotropic correlation tensors with respect to ζ at $\zeta = 0$ and with respect to the isotropy condition with the corresponding operators of the isotropic tensors at $\zeta = 0$, i.e.,

$$(L_{np\ldots t}\overline{u_i u_j \ldots u'_m})_0^* = (L_{np\ldots t} Q_{ij\ldots m})_0. \quad (17)$$

It follows from the general theory of tensor invariants [25] that an nth-rank isotropic correlation tensor can be represented as a linear combination of certain (defining) tensors of the same rank (ζ_i for the first-rank correlation tensor, $\zeta_i\zeta_j$, δ_{ij} for the second-rank correlation tensor, $\zeta_i\zeta_j\zeta_k$, $\delta_{ij}\zeta_k$, $\delta_{ik}\zeta_j$, $\delta_{jk}\zeta_i$ for the third rank tensor, etc.), where the coefficients of these tensors are scalar functions of ζ. Accordingly, conditions (15) and (16) can be satisfied if the number of above-indicated defining tensors is complemented by the corresponding single-point correlations. Thus, proceeding from conditions (15) and (16), we represent the approximate expressions for the anisotropic correlation tensors as a linear combination of tensors which are the defining tensors for the corresponding isotropic tensors and single-point correlations*:

$$\overline{cu'_k} = A(\zeta_s^2)\zeta_i + B(\zeta_s^2)\overline{cu_k},$$

$$\overline{u_i u'_k} = A(\zeta_s^2)\zeta_i\zeta_k + B(\zeta_s^2)\delta_{ik} + c(\zeta_s^2)\overline{u_i u_k}, \quad (18)$$

$$\overline{u_i u_j u'_k} = A(\zeta_s^2)\zeta_i\zeta_j\zeta_k + B(\zeta_s^2)\delta_{ij}\zeta_k + c(\zeta_s^2)(\delta_{ik}\zeta_j + \delta_{jk}\zeta_i) + D(\zeta_s^2)\overline{u_i u_j u_k}$$

etc.

*A similar, though somewhat more complex scheme for the formation of the anisotropic correlation tensors has been proposed by Chou [7]. Here also the notion of axisymmetric turbulence can be used [26, 27].

Inasmuch as the expressions for the correlation tensors are needed only at near points, we expand the coefficients in (18) in a multiple Taylor series:

$$K(\zeta_s^2) = K_0 + \left(\frac{\partial K}{\partial \zeta_m}\right)_0 \zeta_m + \frac{1}{2}\left(\frac{\partial^2 K}{\partial \zeta_m \partial \zeta_n}\right)_0 \zeta_m \zeta_n + \dots$$

Then, using condition (15), we rewrite the tensors (18) in the form

$$\overline{cu_k'} = \overline{cu_k} + \frac{q\pi}{\sqrt{3}}\left[(a_0 + a_m\zeta_m + \dots)\zeta_i + \left(b_m\zeta_m + \frac{1}{2!}b_{mn}\zeta_m\zeta_n + \dots\right)P_{kc}\right],$$

$$\overline{u_iu_k'} = \overline{u_iu_k} + \frac{q^2}{3}\left[(a_0 + a_m\zeta_m + \dots)\zeta_i\zeta_k + \left(b_m\zeta_m + \frac{1}{2!}\right.\right.$$

$$\left.\left.\times\, b_{mn}\zeta_m\zeta_n + \dots\right)\delta_{ik} + \left(c_m\zeta_m + \frac{1}{2!}c_{mn}\zeta_m\zeta_n + \dots\right)R_{ij}\right],$$

$$\overline{u_iu_ju_k'} = \overline{u_iu_ju_k} + \frac{q^3}{3\sqrt{3}}[(a_0 + a_m\zeta_m + \dots)\zeta_i\zeta_j\zeta_k$$

$$+ (b_0 + b_m\zeta_m + \dots)\delta_{ij}\zeta_k + (c_0 + c_m\zeta_m + \dots)(\delta_{ik}\zeta_j + \delta_{jk}\zeta_i) + (d_m\zeta_m + \dots)S_{ijk}]. \tag{19}$$

Owing to the uniformity of the turbulence in the region between the two given closely spaced points, the correlation tensors (19) must satisfy the invariance conditions under reflection from any point in space and under interchanging of the two points, i.e., [24],

$$\overline{cu_k'}(\zeta) = -\overline{c'u_k}(\zeta), \quad \overline{u_iu_j'}(\zeta) = \overline{u_i'u_j}(\zeta),$$

$$\overline{u_iu_ju_k'}(\zeta) = -\overline{u_i'u_j'u_k}(\zeta).$$

These conditions imply that $\overline{cu_k'}$, $u_i'u_j'u_k'$, and the other tensors of odd rank contain only odd powers of ζ, while $\overline{u_iu_j'}$ and the other tensors of even rank contain only even powers of ζ. Moreover, the single-point correlations of odd order are equal to zero. Equations (19) can therefore be rewritten in the form

$$\overline{cu_k'} = \frac{q\pi}{\sqrt{3}}\left[\left(a_0 + \frac{1}{2!}a_{mn}\zeta_m\zeta_n + \dots\right)\zeta_i\right], \tag{20}$$

$$\left.\begin{aligned} \overline{u_iu_k'} = \overline{u_iu_k} + \frac{q^2}{3}\left[\left(a_0 + \frac{1}{2!}a_{mn}\zeta_m\zeta_n + \dots\right)\zeta_i\zeta_k + \left(\frac{1}{2!}b_{mn}\zeta_m\zeta_n\right.\right. \\ \left. + \frac{1}{4!}\beta_{mnjl}\zeta_m\zeta_n\zeta_j\zeta_l + \dots\right)\delta_{ik} + \left(\frac{1}{2!}c_{mn}\zeta_m\zeta_n\right. \\ \left.\left. + \frac{1}{4!}\gamma_{mnjl}\zeta_m\zeta_n\zeta_j\zeta_l + \dots\right)R_{ik}\right], \quad \overline{u_iu_ju_k'} = \frac{q^3}{3\sqrt{3}}\left[(a_0 + \dots)\zeta_i\zeta_j\zeta_k\right. \\ + \left(b_0 + \frac{1}{2!}b_{mn}\zeta_m\zeta_n + \dots\right)\zeta_k\delta_{ij} + \left(c_0 + \frac{1}{2!}c_{mn}\zeta_m\zeta_n + \dots\right) \\ \left.\times\,(\zeta_i\delta_{jk} + \zeta_j\delta_{ik})\right]. \end{aligned}\right\} \tag{21}$$

The following relations are implications of expressions (20)-(21):

$$(\Delta_\zeta \overline{cu_s'})_0 = 0, \quad \left(\frac{\partial}{\partial \zeta_p}\overline{u_mu_n'}\right)_0 = 0, \tag{22}$$

$$\left(\frac{\partial^3}{\partial\zeta_p\partial\zeta_r\partial\zeta_s}\overline{u_mu_n'}\right)_0 = 0, \quad \left(\frac{\partial^2}{\partial\zeta_p\partial\zeta_r}\overline{u_mu_nu_l'}\right)_0 = 0.$$

We now give a more detailed analysis of the even- and odd-rank anisotropic two-point correlation tensor for near points.

If the second-rank tensor (21), written out to fourth powers of ζ, satisfies the incompressibility condition

$$\frac{\partial}{\partial \zeta_k}\overline{u_iu_k'} = 0,$$

after some straightforward manipulations we obtain the relations

$$b_{mn} = -(4a_0\delta_{mn} + c_{ms}R_{sn}),$$

$$\beta_{mnjl} = -3\left(3!a_{jl}\delta_{mn} + \frac{1}{3}\gamma_{mjls}R_{sn}\right).$$

Due to the symmetry of the coefficients b_{mn} and β_{mnjl} under the permutation of indices m and n we obtain

$$c_{mn} = c_0\delta_{mn}, \ \gamma_{mjls} = \gamma_{jl}\delta_{ms}.$$

Consequently, the given tensor can be rewritten in the form

$$\overline{u_iu_k'} = \frac{q^2}{3}\left\{R_{ik} + [a_0(\zeta_i\zeta_k - 2r^2\delta_{ik}) + \frac{1}{2}c_0(r^2R_{ik} - R_{mn}\zeta_m\zeta_n\delta_{ik})] + \left[\frac{1}{2}a_{mn}\zeta_m\zeta_n\left(\zeta_i\zeta_k - \frac{3}{2}r^2\delta_{ik}\right) + \frac{1}{4!}\gamma_{mn}\zeta_m\zeta_n(r^2R_{ik} - R_{jl}\zeta_j\zeta_l\delta_{ik})\right] + \ldots\right\}. \quad (23)$$

From the expression for the second-rank isotropic correlation tensor for near points we infer

$$(\Delta_\zeta Q_{i,k})_0 = -15\overline{u}^2\rho_{i,k}^{(2)*}, \text{ where } \rho_{i,k}^{(2)*} = -\frac{1}{3}\left(\frac{\partial^2 f}{\partial r^2}\right)_0\delta_{ik}, \ \rho_{s,s}^{(2)*} = \frac{1}{\lambda_g^2}.$$

Therefore, bearing condition (17) in mind, we can introduce the following definition of the second-order tensor of microscales of anisotropic two-point correlations:

$$\rho_{i,k}^{(2)} = -\frac{1}{5q^2}(\Delta_\zeta\overline{u_iu_k'})_0. \quad (24)$$

If the tensor (23) satisfies condition (24), we obtain the following relations for the scalar coefficients of the second powers of ζ:

$$\overline{a}_0 = \frac{1}{2}, \ \rho_{i,k}^{(2)} = \frac{1}{3}\rho_{s,s}^{(2)}\delta_{ik} - \frac{1}{5}\rho_{s,s}^{(2)}\overline{c}_0(R_{ik} - \delta_{ik}), \quad (25)$$

in which $\overline{a}_0 = a_0/\rho_{s,s}^{(2)}$, $\overline{c}_0 = c_0/\rho_{s,s}^{(2)}$ are dimensionless scalar coefficients. It is interesting to note that, according to (7) and expression (25), the dissipative function in the double correlation equation (3) has the form

$$2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} = \frac{1}{2}\nu\Delta_x\overline{u_iu_j} - \frac{2}{3}\nu q^2(\alpha - 5)\rho_{s,s}^{(2)}\delta_{ik} + 2\nu\alpha\rho_{s,s}^{(2)}\overline{u_iu_j}, \quad (26)$$

where $\alpha = -3\overline{c}_0$.

Comparing (26) with the analogous expressions derived by Chou in [7] and Rotta in [8], we perceive at once that the latter are applicable only in uniform turbulence. In the present study it is not necessary to treat expression (26) as an approximation [containing the unknown coefficient α] of the correlation (7), because we have the differential equation (9) for the tensor $(\Delta_\zeta\overline{u_iu_j'})_0$. We use expression (25) to determine $\overline{c}_0$. Thus, since (25) holds for any fixed indices $i = \varphi$ and $k = \psi$, the coefficient $\overline{c}_0$ can be represented in the form

$$\overline{c}_0 = 5\left(\frac{1}{3}\delta_{\varphi\psi} - \frac{\rho_{\varphi\psi}^{(2)}}{\rho_{ss}^{(2)}}\right)\frac{1}{R_{\varphi\psi} - \delta_{\varphi\psi}} \quad (27)$$

By analogy with (24), in accordance with condition (17) we readily deduce the relation

$$(\Delta_\zeta\Delta_\zeta\overline{u_iu_k'})_0 = -\frac{35}{3}q^2l_{ik}^{(2)}, \quad (28)$$

where

$$l_{i,k}^{(2)*} = -\frac{1}{3}\left(\frac{\partial^4 f}{\partial r^4}\right)_0\delta_{ik}; \ l_{s,s}^{(2)*} = -\left(\frac{\partial^4 f}{\partial r^4}\right)_0.$$

If the tensor (23) satisfies condition (28), we can derive the following relation between the coefficients a_{mn} and γ_{mn}:

$$a_{mn} = -\frac{35}{8} l_{mn}^{(2)} + \left[\frac{13}{8} l_{ss}^{(2)} + \frac{1}{35} \left(\frac{61}{8} \gamma_{ss} - \frac{1}{3} \gamma_{ps} R_{sp} \right) \right] \delta_{mn} - \frac{5}{24} \gamma_{ss} R_{mn}.$$

Bearing in mind the expressions obtained above for the coefficients, we represent the second-rank correlation tensor (23) in the form

$$\overline{u_i u_k'} = \frac{q^2}{3} \Big\{ R_{ik} + \frac{1}{2} \rho_{ss}^{(2)} [\zeta_i \zeta_k - 2r^2 \delta_{ik} + \bar{c}_0 (r^2 R_{ik} - R_{mn} \zeta_m \zeta_n \delta_{ik})] + \frac{1}{2} \left(\zeta_i \zeta_k - \frac{3}{2} r^2 \delta_{ik} \right) \Big(\Big[\frac{13}{8} l_{ss}^{(2)}$$
$$+ \frac{1}{35} \left(\frac{61}{8} \gamma_{ss} - \frac{1}{3} \gamma_{mn} R_{mn} \right) \Big] r^2 - \frac{5}{8} \left(7 l_{mn}^{(2)} + \frac{1}{3} \gamma_{ss} R_{mn} \right) \zeta_m \zeta_n \Big) + \frac{1}{4!} \gamma_{mn} \zeta_m \zeta_n (r^2 R_{ik} - R_{jl} \zeta_j \zeta_l \delta_{ik}) + \ldots \Big\} \quad (29)$$

where $\bar{c}_0$ is determined by expression (27). It is easily verified that (29) coincides with $Q_{i,k}$ under isotropy.

Using expression (29), we represent the "unknown" functions of the second-rank tensor in Eqs. (9) in the form

$$\left(\frac{\partial^2}{\partial \zeta_m \partial \zeta_n} \overline{u_i u_j'} \right)_0 = \frac{1}{6} q^2 \rho_{s,s}^{(2)} F_{mn}^{ij},$$
$$F_{mn}^{ij} = \delta_{im} \delta_{jn} + \delta_{in} \delta_{jm} - 4\delta_{mn} \delta_{ij} + 2\bar{c}_0 (R_{ij} \delta_{mn} - R_{mn} \delta_{ij}). \quad (30)$$

We next consider the third-rank tensor. If the tensor (21) satisfies the incompressibility condition

$$\frac{\partial}{\partial \zeta_k} \overline{u_i u_j u_k'} = 0,$$

we can derive the following relations for the coefficients:

$$b_0 = -\frac{2}{3} c_0, \quad b_{mn} \delta_{ij} = -\frac{2}{5} (5 a_0 \delta_{mi} \delta_{jn} + c_{mj} \delta_{in} + c_{mi} \delta_{jn} + c_{mn} \delta_{ij}).$$

Taking these relations into account, we write the tensor (21) in the form

$$\overline{u_i u_j u_k'} = \frac{q^3}{3\sqrt{3}} \Big\{ c_0 \left(\zeta_i \delta_{jk} + \zeta_j \delta_{ik} - \frac{2}{3} \zeta_k \delta_{ij} \right)$$
$$+ c_{mn} \zeta_m \zeta_n \left[\frac{1}{2} (\zeta_i \delta_{jk} + \zeta_j \delta_{ik}) - \frac{1}{5} \zeta_k \delta_{ij} \right] - \frac{1}{5} (c_{mi} \zeta_j + c_{mj} \zeta_i) \zeta_m \zeta_n + \ldots \Big\}. \quad (31)$$

From the isotropic triple correlation expression we obtain [24]

$$\left(\frac{\partial}{\partial \zeta_j} \Delta_\zeta Q_{ij,k} \right)_0 = -35 (\overline{u^2})^{3/2} \rho_{i,k}^{(3)*},$$

where

$$\rho_{i,k}^{(3)*} = \frac{1}{3} \left(\frac{\partial^3 h}{\partial r^3} \right)_0 \delta_{ik}, \quad \rho_{s,s}^{(3)*} = \left(\frac{\partial^3 h}{\partial r^3} \right)_0.$$

Consequently, bearing condition (17) in mind, we define the third-order tensor of microscales of anisotropic correlations:

$$\rho_{i,k}^{(3)} = -\frac{3! \sqrt{3}}{70 q^3} \left(\frac{\partial}{\partial \zeta_j} \Delta_\xi \overline{u_i u_j u_k'} \right)_0. \quad (32)$$

If (31) satisfies condition (32), we obtain

$$c_{mn} = -\frac{5}{8} \left(35 \rho_{m,n}^{(3)} - 13 \rho_{s,s}^{(3)} \delta_{mn} \right), \quad c_{ss} = -\frac{5}{2} \rho_{s,s}^{(3)}.$$

Also, we can readily show that*

$$c_0 = \frac{3\sqrt{3}}{10 q^3} \left(\frac{\partial}{\partial \zeta_i} \overline{u_i u_k u_k'} \right)_0 = 0.$$

*It can be shown analogously that any two-point correlation tensor of odd rank does not contain first powers of ζ.

Thus, the expression for the third-rank anisotropic correlation tensor for near points is written in the form

$$\overline{u_i u_j u'_k} = \frac{q^3}{3\sqrt{3}} \cdot \frac{5}{8} \left\{ \left(35\, \rho^{(3)}_{m,n} \zeta_m \zeta_n - 13\rho^{(3)}_{s,s} r^2\right) \left[\frac{1}{2} (\zeta_i \delta_{jk} + \zeta_j \delta_{ik}) - \frac{1}{5} \zeta_k \delta_{ij} \right] \right.$$

$$\left. - \frac{1}{5} \left[35 \left(\rho^{(3)}_{m,j} \zeta_m \zeta_i + \rho^{(3)}_{m,i} \zeta_m \zeta_j\right) - 26\rho^{(3)}_{s,s} \zeta_i \zeta_j \right] \zeta_k + \ldots \right\}. \tag{33}$$

Using expression (33), we represent the "unknown" functions of the third-rank tensor in Eqs. (9) and (12) in the form*

$$\left[\frac{\partial}{\partial \zeta_k} \Delta_\zeta \left(\overline{u_i u_k u'_j} + \overline{u_j u_k u'_i}\right) \right]_0 = -\frac{70}{3\sqrt{3}}\, q^3 \rho^{(3)}_{ij}, \tag{34}$$

$$-\frac{1}{\rho} \left(\frac{\partial}{\partial \zeta_j} \overline{p u'_i} \right)_0 = \left(\frac{\partial^3}{\partial \zeta_j \partial \zeta_m \partial \zeta_n} \overline{u_m u_n u'_i} \right)_0 = \frac{35}{2\sqrt{3}}\, q^3 \left(3\rho^{(3)}_{i,j} - \rho^{(3)}_{s,s} \delta_{ij}\right), \tag{35}$$

$$\left(\frac{\partial}{\partial \zeta_r} \overline{u_m u_n u'_s} \right)_0 = 0. \tag{36}$$

We estimate the unknown function $(\overline{\Delta_\zeta p u'_r u'_s})_0$ appearing in Eq. (14) for uniform turbulence:

$$\frac{1}{\rho} \left(\Delta_\zeta \overline{p u'_r u'_s}\right)_0 = -\left(\frac{\partial^2}{\partial \zeta_m \partial \zeta_n} \overline{u_m u_n u'_r u'_s} \right)_0. \tag{37}$$

We express the second-rank tensor of triple-correlation microscales $\rho^{(3)}_{i,j}$ and the tensor $l^{(2)}_{i,j}$ in terms of the tensor of double-correlation microscales $\rho^{(2)}_{i,j}$, for which we have a descriptive differential equation. It can be shown that the following relations hold for isotropy:

$$\rho^{(3)*}_{i,j} = -\frac{1}{2} S^* \rho^{(2)*}_{i,j} \left(\rho^{(2)*}_{s,s} \right)^{1/2}, \quad l^{(2)*}_{i,j} = -\frac{1}{2\nu} S^*_\nu \left(\overline{u^2}\right)^{\frac{1}{2}} \rho^{(2)*}_{i,j} \left(\rho^{(2)*}_{s,s} \right)^{1/2}, \tag{38}$$

in which $S^* = \overline{(\partial u_r/\partial x_r)^3}/[\overline{(\partial u_r/\partial x_r)^2}]^{3/2}$ is the asymmetry coefficient of the density function for the probability of the velocity derivatives [2] and $S^*_\nu = 2\nu\, \overline{(\partial^2 u_r/\partial x_r^2)^2}/[\overline{(\partial u_r/\partial x_r)^2}]^{3/2}$ is a dimensionless coefficient composed of mixed moments, i.e., the first and second derivatives of the velocity fluctuations [28].

Bearing in mind condition (17) and relation (38), we obtain for anisotropy

$$\rho^{(3)}_{i,j} = -\frac{1}{2} S \rho^{(2)}_{i,j} \left(\rho^{(2)}_{s,s} \right)^{1/2},$$

$$l^{(2)}_{i,j} = -\frac{1}{2\sqrt{3}} \cdot \frac{1}{\nu}\, q S_\nu \rho^{(2)}_{i,j} \left(\rho^{(2)}_{s,s} \right)^{1/2},$$

$$S = \frac{6}{7}\sqrt{15} \frac{\left(\frac{\partial}{\partial \xi_j} \Delta_\xi \overline{u_j u_s u'_s} \right)_0}{\left(-\Delta_\xi \overline{u_s u'_s}\right)_0^{3/2}}, \quad S_\nu = \frac{6}{7}\sqrt{15}\, \nu \frac{\left(\Delta_\xi \Delta_\xi \overline{u_s u'_s} \right)_0}{\left(-\Delta_\xi \overline{u_s u'_s}\right)_0^{3/2}}.$$

Taking relations (5), (7), (10), (13), (22), (24), (28), (30), and (34)-(39) into account, we write Eqs. (3), (4), (9), (12), and (14) in the form†

$$\frac{\partial}{\partial \tau} \overline{u_i u_j} + U_k \frac{\partial}{\partial x_k} \overline{u_i u_j} + \overline{u_i u_k} \frac{\partial U_j}{\partial x_k} + \overline{u_j u_k} \frac{\partial U_i}{\partial x_k} - \frac{1}{2} \nu \Delta_x \overline{u_i u_j}$$

$$+ 10\nu \overline{\rho}^{(2)}_{i,j} + \frac{1}{2\rho} \left(\frac{\partial}{\partial x_j} \overline{p u_i} + \frac{\partial}{\partial x_i} \overline{p u_j} \right) + \frac{\partial}{\partial x_k} \overline{u_i u_j u_k} = 0,$$

*The left-hand side of (35) is obtained by differentiation of (12) with respect to ζ_j at $\zeta = 0$.

†We make use of the obvious relation

$$\left[\left(\frac{\partial^n}{\partial x_p \partial x_r \ldots \partial x_t} \right)_{AB} \overline{u_i u_j \ldots u'_m} \right]_0 = \frac{\partial^n}{\partial x_p \partial x_r \ldots \partial x_t} \overline{u_i u_j \ldots u'_m}.$$

$$\frac{\partial}{\partial \tau}\overline{u_iu_ju_k}+U_l\frac{\partial}{\partial x_l}\overline{u_iu_ju_k}+\overline{u_iu_ju_l}\frac{\partial U_k}{\partial x_l}+\overline{u_ju_ku_l}\frac{\partial U_i}{\partial x_l}$$

$$+\overline{u_iu_ku_l}\frac{\partial U_j}{\partial x_l}-\frac{1}{4}\nu\Delta_x\overline{u_iu_ju_k}+\overline{u_ku_l}\frac{\partial}{\partial x_l}\overline{u_iu_j}+\overline{u_ju_l}\frac{\partial}{\partial x_l}\overline{u_iu_k}$$

$$+\overline{u_iu_l}\frac{\partial}{\partial x_l}\overline{u_ju_k}+\frac{1}{2\rho}\left(\frac{\partial}{\partial x_i}\overline{pu_ju_k}+\frac{\partial}{\partial x_j}\overline{pu_iu_k}+\frac{\partial}{\partial x_k}\overline{pu_iu_j}\right)=0,$$

$$\frac{1}{4\rho}\Delta_x\overline{pu_r}+\frac{\partial U_m}{\partial x_n}\cdot\frac{\partial}{\partial x_m}\overline{u_nu_r}+\frac{1}{4}\frac{\partial^2}{\partial x_m\partial x_n}\overline{u_mu_nu_r}=0,$$

$$\frac{1}{4\rho}\Delta_x\overline{pu_ru_s}+\frac{\partial U_m}{\partial x_n}\cdot\frac{\partial}{\partial x_m}\overline{u_nu_ru_s}$$

$$+\frac{1}{4}\cdot\frac{\partial^2}{\partial x_m\partial x_n}\left(\overline{u_mu_n}\cdot\overline{u_ku_l}+\overline{u_mu_k}\cdot\overline{u_nu_l}+\overline{u_mu_l}\cdot\overline{u_nu_k}\right)=0,$$

$$\frac{\partial}{\partial \tau}\bar{\rho}_{i,j}+U_k\frac{\partial}{\partial x_k}\bar{\rho}_{i,j}-\frac{1}{20}\left(\overline{u_iu_k}\frac{\partial}{\partial x_k}\Delta_xU_j+\overline{u_ju_k}\frac{\partial}{\partial x_k}\Delta U_i\right)$$

$$-\frac{1}{20}\Delta_xU_k\frac{\partial}{\partial x_k}\overline{u_iu_j}-\frac{1}{15}\bar{\rho}_{s,s}F^{ij}_{lk}\frac{\partial U_k}{\partial x_l}+\bar{\rho}_{i,k}\frac{\partial U_j}{\partial x_k}$$

$$+\bar{\rho}_{j,k}\frac{\partial U_i}{\partial x_k}+\frac{77}{6\sqrt{3}}S\bar{\rho}_{i,j}\bar{\rho}_{s,s}^{-\frac{1}{2}}-\frac{7}{2\sqrt{3}}S\bar{\rho}_{s,s}^{-\frac{3}{2}}\delta_{ij}$$

$$-\frac{1}{2}\nu\Delta_x\bar{\rho}_{i,j}+\frac{7}{3\sqrt{3}}S_\nu\bar{\rho}_{i,j}\bar{\rho}_{s,s}^{-\frac{1}{2}}=0\quad\left(\bar{\rho}_{i,j}=q^2\rho^{(2)}_{i,j}\right).$$

The foregoing equations in combination with Eqs. (1)-(2) form a closed system of equations describing the dynamics of nonuniform turbulence. This system includes two statistical coefficients, S and S_ν, which are directly measurable statistical characteristics of the velocity fluctuation field. The indicated coefficients have been investigated in adequate detail both theoretically and experimentally for the isotropic case [28-31]. The remarkable property of these coefficients is their conservatism under variation of the turbulent Reynolds number (for $R^*_\rho > 100$), whereby the coefficients may be regarded as universal constants ($S^* \simeq -0.4$; $S^*_\nu = 0.6$). The conservatism of the asymmetry coefficient with respect to the turbulent Reynolds number (for $R_\rho > 150$, corresponding to a value of the universal coordinate $x_2v_*/\nu \simeq 80$) has been demonstrated experimentally for nonuniform turbulence [23], where the characteristic S varies between the limits −0.3 to −0.4, depending on the average-flow Reynolds number. Consequently, the values of S determined for isotropy and anisotropy practically coincide. Insofar as the authors are aware, the coefficient S_ν has never been measured for nonuniform turbulence. Therefore, the approximate value of this coefficient is $S_\nu = 0.6$, as determined "theoretically" by means of the Heisenberg hypothesis on the spectral transfer of energy and as confirmed experimentally [28].

NOTATION

x_i	are the rectangular coordinates (i = 1, 2, 3);
ν	is the kinematic viscosity;
τ	is the time;
ρ	is the density;
Δ_x	is the Laplace operator on the variable x;
U_i	is the average flow velocity;
u_i	is the velocity fluctuation;
P	is the average pressure;
p	is the pressure fluctuation;
$(u_i)_A$	fluctuation at point A;
$(u_i)_B = u_i'$	is the fluctuation at point B;
$[F(\zeta)]_0$	is the function evaluated at the point $\zeta = 0$;

c	is the scalar substance;
$q = (\overline{u_i u_i})^{1/2}$;	
$r^2 = \xi_i^2$;	
$\pi = (\overline{c^2})^{1/2}$;	
L	is the differential operator symbol;
Δ_ξ	is the Laplace operator on the variable ξ;
$P_{kc} = \sqrt{3}(\overline{cu_k})/q\pi$;	
$R_{ij} = 3(\overline{u_i u_j})/q^2$;	
$S_{ijk} = 3\sqrt{3}(\overline{u_i u_j u_k})/q^3$	are single-point correlation coefficients;
$(f)^*$	is the symbol for a function in the case of isotropy;
$f(r) = \overline{u_r u_r'}/\overline{u^2}$	is the longitudinal double correlation coefficient;
$h(r) = \overline{u_n u_n u_r'}/(\overline{u}^2)^{3/2}$	is the triple correlation coefficient;
λ_g	is the transverse scale of isotropic turbulence;
R_ρ $\sqrt{\overline{u}_i^2}\nu(\rho_{ss}^{(2)})^{2/2}$	is the turbulent Reynolds number;
$R_\rho = v_* = \sqrt{\tau_w/\rho}$	is the dynamic velocity.

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